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Abstract: A $(d,1)$-total labelling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least d in absolute value. The *span* of a $(d,1)$-total labelling is the maximum difference between two labels. The minimum span of a $(d,1)$-total labelling of G is denoted by $\lambda_d^T(G)$.

We show $\lambda_d^T \leq 2\Delta + d - 1$ and conjecture $\lambda_d^T \leq \Delta + 2d - 1$, where Δ is the maximum degree of a vertex in a graph. We prove this conjecture for complete graphs. More precisely, we determine the exact value of $\lambda_d(K_n)$ except for even n in the interval $[d+5, 6d^2 - 10d + 4]$ for which we show that $\lambda_d^T(K_n) \in \{n + 2d - 3, n + 2d - 2\}$.

We then give some evidences for the conjecture to be true. We prove it when $\Delta \leq 3$. We also show that as $n = |G| \to \infty$, $\lambda_d^T \leq \Delta + O(\log n / \log\log n)$ and the proportion of graphs on vertices $1, 2, \ldots, n$ with $\lambda_d^T > \Delta + 2d - 1$ is very small. Finally, we show that any vertex colouring may be extended to a fractional $(d,1)$-total labelling with span at most $\Delta + 3d$.

Key-words: total colouring

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Coloration (d,1)-totale de graphes

Résumé : Une coloration $(d,1)$-totale d'un graphe G est une application de $V(G) \cup E(G)$ dans l'ensemble des entiers telle que : (i) deux sommets adjacents ont des images différentes, (ii) deux arêtes adjacentes ont des images différentes, et (iii) les images d'un sommet et d'une arête adjacente diffère d'au moins d en valeur absolue. L'*écart* d'une coloration $(d,1)$-totale est la différence maximale entre deux images par celle-ci. L'écart minimum pour une coloration $(d,1)$-totale est noté $\lambda_d^T(G)$.

Nous montrons $\lambda_d^T \leq 2\Delta + d - 1$ et nous conjecturons $\lambda_d^T \leq \Delta + 2d - 1$, avec Δ le degré maximal d'un sommet du graphe. Nous prouvons cette conjecture pour les graphes complets. Plus précisément, nous déterminons la valeur exacte de $\lambda_d(K_n)$ excepté pour les n pairs dans l'intervalles $[d+5, 6d^2-10d+4]$ pour lesquels nous montrons $\lambda_d^T(K_n) \in \{n+2d-3, n+2d-2\}$.

Nous donnons ensuite des résultats qui laissent penser que cette conjecture est vraie. Tout d'abord nous prouvons celle-ci lorsque $\Delta \leq 3$. Nous montrons également que $\lambda_d^T \leq \Delta + O(\log n / \log\log n)$ lorsque $n = |G| \to \infty$ et que la proportion de graphes sur les sommets $1, 2, \ldots, n$ pour lesquels $\lambda_d^T > \Delta + 2d - 1$ est très petite. Enfin, nous montrons que toute coloration des sommets peut être étendue en une coloration $(d,1)$-totale fractionnaire d'écart au plus $\Delta + 3d$.

Mots-clés : coloration totale

1 Introduction

In the channel assignment problem, the following situation occurs : we need to assign radio frequency bands to transmitters (each station gets one channel which corresponds to an integer). In order to avoid interference, if two stations are too close, then the separation of the channels assigned to them has to be at least two. Moreover, if two stations are close (but not too close), then they must receive different channels. Motivated by this problem, Griggs and Yeh [3] introduced $L(2,1)$-labellings. Its natural generalisation $L(d,1)$-labellings of a graph G is an integer assignment L to the vertex set $V(G)$ such that:

$|L(u)-L(v)| \geq d$ if $d_G(u,v)=1$ and $|L(u)-L(v)| \geq 1$ if $d_G(u,v)=2$.

There are several articles studying this labelling. In [1] it was studied for chordal graphs. In particular, Whittlesey, Georges and Mauro [7] studied L(2,1)-labellings of first subdivision of a graph G. The *first subdivision* of a graph G is the graph $s_1(G)$ obtained from G by inserting one vertex along each edge of G. A L(d,1)-labelling of $s_1(G)$ corresponds to an assignment of integers to $V(G) \cup E(G)$ such that:

(i) any two adjacent vertices of G receive distinct integers,

(ii) any two adjacent edges of G receive distinct integers, and

(iii) a vertex and an edge incident receive integers that differ by at least d in absolute value.

We call such an assignment *(d,1)-total labelling of* G. It is a total colouring strenghtened with an extra condition insisting on a minimal separation of d between incident vertices and edges.

The *span* of an L(d,1)-labelling is the maximum difference between two labels. Analogously, the *span* of a (d,1)-total labelling is the maximum difference between two labels. The $(d,1)$-*total number* of a graph G, denoted by $\lambda_d^T(G)$, is the minimum span of a $(d,1)$-total labelling of G. Note that a $(1,1)$-total labelling is a total colouring and that $\lambda_1^T = \chi^T - 1$ where χ^T is the total chromatic number.

The aim of this paper is to study $(d,1)$-total labellings of graphs and in particular, bounds for the $(d,1)$-total number λ_d^T as a function of the maximal degree Δ of the graph.

In the first section, we give some general upper bounds and show that $\lambda_d^T \leq 2\Delta + d - 1$. However this upper bound is not tight for graphs with large maximal degree. We conjecture that $\lambda_d^T \leq \Delta + 2d - 1$ and so $\lambda_d^T \leq \min\{\Delta+2d-1, 2\Delta+d-1\}$. For $d=1$, this conjecture is the Total Colouring Conjecture $\chi^T \leq \Delta + 2$. Molloy and Reed [6] proved that there is a constant c such that the total chromatic number is at most $\Delta + c$. They proved that $c \leq 10^{26}$ [6]. According to Bruce Reed, a similar proof would give an analogous theorem for $(d,1)$-total labelling but with a larger constant c_d.

We prove that this conjecture holds for complete graphs. In section 3, we give more precise results by determining the exact value of the $(d,1)$-total number of almost all complete graphs : If n is odd then $\lambda_d(K_n) = \min\{n+2d-3, 2n+d-3\}$; if n is even then $\lambda_d(K_n) = \min\{n+2d-3, 2n+d-3\}$ if $n \leq d+5$, $\lambda_d^T(K_n) = n+2d-2$ if $n > 6d^2-10d+4$ and $\lambda_d^T(K_n) \in \{n+2d-3, n+2d-2\}$ otherwise.

In Section 4 some evidences are provided to support the Conjecture 2.1. We examine g the cases when Δ is small. We first give the $(d,1)$-total number of graphs with maximum degree 2. We then prove the conjecture for graphs with maximum degree at most 3. We also show that the upper bound 8 of Conjecture 2.1 when $d = \Delta = 3$ is not tight: $\lambda_3^T \leq 7$ if $\Delta \leq 3$. At last we establish that if $\Delta \leq 4$ then $\lambda_2^T(G) \leq 8$.

In the last section, we give some more evidences for Conjecture 2.1 to be true. We consider the technique of extending a vertex-colouring into a $(d,1)$-total labelling. We first extend the results of [5] on total colouring to $(d,1)$-total labellings. We show that as $n = |G| \to \infty$, $\lambda_d^T \leq \Delta + O(\log n/\log\log n)$ and the proportion of graphs on vertices $1, 2, \dots, n$ with $(d,1)$-total number bigger than $\Delta + 2d - 1$ is very small. Finally, we show that any vertex colouring may be extended to a fractional $(d,1)$-total labelling with span at most $\Delta + 3d$.

2 Some bounds

Looking to the labels of a vertex of maximal degree and its incident edges, it is easy to see that $\lambda_d^T \geq \Delta + d - 1$. This lower bound may be increased sometimes :

Proposition 2.1 *(i) If G is Δ-regular then $\lambda_d^T \geq \Delta + d$.*

(ii) If $d \geq \Delta$, then $\lambda_d^T \geq \Delta + d$.

Proof. Suppose that that G admits a $(d,1)$-labelling in $[0, \Delta + d - 1]$. Then every vertex must be labelled 0 or $\Delta + d - 1$. Let v be a vertex of G. Without loss of generality, we may suppose that v is labelled 0. Then its incident edges are labelled with $\{d, d+1, \dots, \Delta+d-1\}$.

(i) Let vw be the edge that is labelled $\Delta + d - 1$. Then v cannot be labelled $\Delta + d - 1$ nor 0. This is contradiction.

(ii)Let vw be the edge that is labelled d. The vertex w must have a label that is bigger than $2d - 1$, thus bigger than $\Delta + d - 1$. This is contradiction. ■

Proposition 2.2

$$\lambda_d^T \leq \chi + \chi' + d - 2$$

$$\lambda_d^T \leq 2\Delta + d - 1$$

Proof. Let c be a vertex colouring of G with the χ integers of $[0, \chi - 1]$ and c' be an edge colouring of G with the χ' integers of $[\chi - 1 + d, \chi + \chi' + d - 2]$. Then the union of c and c' is obviously a $(d,1)$-labelling of G. Thus $\lambda_d^T \leq \chi + \chi' + d - 2$.

If G is neither a complete graph nor an odd cycle then $\chi \leq \Delta + 1$ by Brook's theorem and $\chi' \leq \Delta + 1$ by Vizing's theorem. Hence, $\lambda_d^T \leq 2\Delta + d - 1$.

Suppose now that G is the complete graph K_n on n vertices. $\chi = n = \Delta + 1$. If n is even then $\chi' = \Delta$. So $\lambda_d^T(K_n) \leq 2\Delta + d - 1$. If n is odd, then $\chi' = \Delta + 1$. Let c' be an edge colouring of G with n colours. And let M_i, $1 \leq i \leq n$, be the matchings corresponding to the colour classes. Each vertex is in every M_i but one and each M_i contains all the vertices but one v_i. For $1 \leq i \leq n$, label the vertex v_i with $n - i$ and the edges of M_i with $n + d - 3 + i$.

Since v_1 is not incident to any edge of M_1, then we have a $(d,1)$-total labelling of K_n in $[0, 2n+d-3] = [0, 2\Delta+d-1]$.

At last suppose that G is an odd cycle. Label the vertices with 0, 1 and 2 such that exactly one vertex v is assigned 2. And label the edges with 3, 4 and 5 such that exactly one edge e, not incident to v, is assigned 3. ■

Corollary 2.1 *If G is bipartite then $\Delta + d - 1 \leq \lambda_d^T \leq \Delta + d$.*
In particular, if $d \geq \Delta$ or G is regular then $\lambda_d^T = \Delta + d$.

Proof. If G is bipartite then $\chi = 2$ and $\chi' = \Delta$ by König's Theorem. Then Propositions 2.2 and 2.1 give the result. ■

If $d < \Delta$, there are bipartite graphs for which $\lambda_d^T = \Delta + d - 1$ and graphs for which $\lambda_d^T = \Delta + d$ and Havet and Thomassé [4] proved that it is NP-complete to decide the exact value for a bipartite graph G.

If $d \geq \Delta$, then the upper bound $2\Delta + d - 1$ of Proposition 2.2 is attained for the complete graphs :

Proposition 2.3 *If $d \geq n+1$ then $\lambda_d^T(K_n) = 2n + d - 3$.*

Proof. By Proposition 2.2, $\lambda_d^T(K_n) \leq 2n + d - 3$.

Suppose for a contradiction, that K_n admits a $(d,1)$-total labelling in $[0, 2n+d-4]$. The vertices must be labelled with labels in $[0, n-2] \cup [n+d-2, 2n+d-4]$. Indeed, for a vertex labelled in $[n-1, n+d-1]$, at most $2n+d-3-n-d+1 = n-2$ labels are available for its incident edges and this is a contradiction.

Let i (resp. j) be the largest integer in $[0, n-2]$ such that a vertex is labelled i (resp. $2n+d-4-j$). Since n different labels are used for the vertices, $i + j + 2 \geq n$. Let us now consider the label l of the edge joining the vertices labelled i and $2n+d-4-j$. We have $d + i \leq l \leq 2n - 4 - j$. Hence $d \leq n - 2$ which is a contradiction. ■

However, if $d \leq n$, the upper bound $2\Delta + d - 1$ of Proposition 2.2 is far from being tight for graph with large maximal degree.

Proposition 2.4 *Let G be a graph on n vertices, then $\lambda_d^T(G) \leq n + 2d - 2$.*

Proof. Assign to each vertex v a different integer $l(v)$ of $[0, n-1]$ and assign to an edge uv the integer $l(u) + l(v) + d \mod n + 2d - 1$. We show that l is a (d,1)-total labelling of G. Two adjacent edges have different labels since two distinct vertices have different labels. And clearly, $|l(uv) - l(u) \mod n + 2d - 1| \geq d$. Thus l is a (d,1)-total labelling. ■

For graphs with maximal degree $\Delta = n - 1$, Proposition 2.4 yields $\lambda_d^T \leq \Delta + 2d - 1$. Such an upper bounds seems to be the accurate one. Hence, we conjecture the following :

Conjecture 2.1

$$\lambda_d^T \leq \Delta + 2d - 1$$

so
$$\lambda_d^T \leq \min\{\Delta + 2d - 1, 2\Delta + d - 1\}$$

3 (d,1)-total labellings of complete graphs

Proposition 3.1 *If $n \geq d$, then $\lambda_d^T(K_n) \geq n + 2d - 3$.*

Proof. Suppose that there is a $(d,1)$-total labelling with labels in $[0, n + 2d - 4]$. Let l be a label in $[d - 1, n + d - 3]$. A vertex cannot be labelled l since there are at most $|[0, n + 2d - 4] \setminus [l - d + 1, l + d - 1]| = n - 2$ labels allowed for its $n - 1$ incident edges. Hence only the $2d - 2$ vertices of $[0, d - 2] \cup [n + d - 2, n + 2d - 4]$ may be labelled. (In particular, $n \leq 2d - 2$.) Since $n \geq d$, a vertex must get a label in $[0, d - 2]$ and another one a label in $[n + d - 2, n + 2d - 4]$. Let j_1 be the largest integer in $[0, d - 2]$ labelling a vertex x and $n + 2d - 4 - j_2$ be the smallest integer in $[n + d - 2, n + 2d - 4]$ labelling a vertex y. The edge xy must be labelled in $[j_1 + d, n + d - 4 - j_2]$. Thus $n + d - 4 - j_2 \geq j_1 + d$, so $n \geq j_1 + j_2 + 4$. But the labels of all vertices are in $[0, j_1] \cup [n + 2d - 4 - j_2, n + 2d - 4]$. Hence $n \leq j_1 + j_2 + 2$ which is a contradiction. ■

Remark 3.1 If $n \leq d-1$ then by Theorem 2.2, $\lambda_d^T(K_n) \leq 2\Delta + d - 1 = 2n + d - 3 \leq n + 2d - 4$.

Proposition 2.4 and Proposition 3.1 show that $n + 2d - 3 \leq \lambda_d^T(K_n) \leq n + 2d - 2$ when $d \leq n$. In the rest of the section, we establish the exact value of $\lambda_d^T(K_n)$ between $n + 2d - 3$ and $n + 2d - 2$ for most of the complete graphs.

3.1 Odd complete graphs

Theorem 3.1 *If n is odd then $\lambda_d^T(K_n) \leq n + 2d - 3$.*

Proof. We will present a labelling using the integers in the interval $[-(n - 3)/2 - d, (n - 3)/2 + d]$ as the labels.

Consider K_n, where $V(K_n) = [-(n - 3)/2 - d, -d] \cup \{0\} \cup [d, (n - 3)/2 - d]$, which are also the labels of the vertices. Let $F = \{(i, -i), i \in [d, (n - 3)/2 + d]\}$. We use 0 to label all edges of F.

Before we assign labels to the remaining edges, we partition $K_n - F$ into two isomorphic subgraphs G_1 and G_2. Furthermore, $G_1 = A_1 \bigcup B_1$, where A_1 is a complete graph on $(n + 1)/2$ vertices with the vertex set $[-(n - 3)/2 - d, -d] \cup \{0\}$ and B_1 is a bipartite graph with bipartitions $[d, (n - 3)/2 + d]$ and $[-(n - 3)/2 - d, -d]$ and the edge set $\{(i, j) : i + j \in [1, (n - 3)/2]\}$.

Clearly, G_2 can be considered as the union of A_2 and B_2 and they are isomorphic to A_1 and B_1, respectively. We will label the edges of $K_n - F$ in a symmetric manner in the sense that if an edge e in G_1 receives the label i, then the corresponding edge in G_2 receives the label $-i$.

Notice that the edges of B_1 are also incident with the vertices in G_2. Therefore, the labels used for the edges of B_1 will not be used in G_2. It is clear that only the vertex 0 is in both G_1 and G_2 and our labelling strategy for G_1 is to assign not only distinct labels to it, but also make sure that if p is a label incident with 0, then $-p$ will not. Then with symmetric manner of the labelling, we will extend the labelling to G_2 and obtain a valid one K_n.

In G_1, we label the edges of B_1 first. Notice that the edges of B_1 can be partitioned into $(n-3)/2$ matchings, $M_i, 1 \leq i \leq (n-3)/2$, where $M_i = \{(p,q) : p+q = (n-1)/2 - i\}$. Hence, $|M_i| = i$.

We assign the labels to the edges of B_1 as follows: the edge (i,j) is labelled $i+j$. As we know that $|i|, |j| \geq d$, this assignment does not violate the labelling restriction.

For assigning the labels to the edges of A_1, we consider two cases.

Case 1 : n = 3 (mod 4): Let $n = 4k+3$. Consider K_{2k+2}, where $V(K_{2k+2}) = \{0, -d, -(d+1), ..., -(d+2k)\}$. The edges of K_{2k+2} are labelled as follows :

- If $n < 2d+3$ (or $2k < d$), then we take any 1-factorization of K_{2k+2} and assign the labels $d, d+1, ..., 2k+d$ to the $2k+1$ 1-factors (one label for each 1-factor). The labelling is valid because the labels used for M_i's are $1, 2, ..., 2k$.
- Otherwise, take a 1-factorization $\{F_1, F_2, ..., F_{2k+1}\}$ of K_{2k+2} as described in Lemma 3.1. We use the labels $d, d+1, ..., d+2k$ for the $2k+1$ 1-factors (one label each) as follows. The edges of F_i are labelled $2k-i+1$ for $1 \leq i \leq 2k-d+1$. For the rest we divide into two subcases.

  case 1.1 : d is even. For $0 \leq i \leq (d-2)/2$, F_{2k-2i} is labelled $2k+d-2i$. Assign the rest labels to the unlabelled 1-factors. Clearly, this labelling is not a valid one as it is in conflict with the labels of M_i (the matchings in B_1) and may not be extended in a symmetric manner to G_2. The vertex $-(d+2k)+j$ is incident to edges labelled $1, 2, \ldots, j$ in B_1. Therefore the edges labelled $1, 2, \ldots, j$ incident to it in A_1 must be relabelled.

  (a) For $1 \leq i \leq 2k-d+1$, in F_i, the edges with both endvertices in $\{0\} \cup (\bigcup_{j=d}^{d+i-1} \{-j\})$ are relabelled $-(2d-i+1)$.

  Moreover, to be sure that at most one of the two integers p and $-p$ are used for the edges of A_1 incident to 0, some other edges must be relabelled:

(b1) If $k+1 \geq d$, for $0 \leq i \leq (d-2)/2$, in F_{2k-2i}, reassign the label $-(2d-2-2i)$ to the edge $(0, -(2k+d-i))$.

(b2) If $k+1 < d$, for $0 \leq i \leq k-d/2$, in F_{2k-2i}, reassign the label $-(2k-2i)$ to the edge $(0, -(2k+d-i))$.

We can check that all the edges incident 0 have different labels and if p is one of the labels, then $-p$ is not. Indeed before the relabelling, the labels for the edges incident 0 are $d, d+1, ..., d+2k$. After the relabelling,

- if $k+1 \geq d$, then they are $-(2k+d), -(2k+d-2), \dots, -2d$ (those relabelled with (a)), $-(2d-2), -(2d-4), \dots, -d$ (those relabelled with (b1)), and $d+1, d+3, ..., 2k+3, ..., 2k+d-1$ (the non-relabelled ones);
- if $k+1 < d$, the labels are: $-(2k+d), -(2k+d-2), \dots, -2d$ (those relabelled with (a)), $-2k, -2k+2, \dots, -d$ (those relabelled with (b2)), $2k+2, 2k+4, \dots, 2d-2$, and $d+1, d+3, \dots, 2k+d-1$ (the non-relabelled ones).

Therefore, the labelling we have for G_1 is valid. Then we assign labels to G_2 in a symmetric manner as described before and we will have a valid labelling we want.

case 1.2 : d is odd. For $0 \leq i \leq (d-3)/2$, F_{2k-2i} is labelled $2k+d-1-2i$. Assign the rest labels to the unlabelled 1-factors. We will again adjust the labels for some of the edges as follows.

(a) For $1 \leq i \leq 2k-d+1$, in F_i, the edges with both endvertices in $\{0\} \cup (\bigcup_{j=d}^{d+i-1}\{-j\})$ are reassigned the label $-(2d-i+1)$.

(b1) If $k+1 \geq d$, for $0 \leq i \leq (d-3)/2$, in F_{2k-2i}, reassign the label $-(2d-2-2i)$ to the edge $(0, -(2k+d-i))$.

(b2) If $k+1 < d$, for $0 \leq i \leq k-(d+1)/2$, in F_{2k-2i}, reassign the label $-(2k-2i)$ to the edge $(0, -(2k+d-i))$.

We can verify as before that this labelling is indeed valid.

Case 2 : n = 1 (mod 4): Let $n = 4k+1$. Consider K_{2k+1}, where $V(K_{2k+1}) = \{0, -d, -(d+1), ..., -(2k+d-1)\}$. We label the edges of K_{2k+1} as follows.

- If $n \leq 2d-1$ (or $2k \leq d-1$), then we take any near 1-factorization of K_{2k+1} and assign the labels $d-1, d, ..., 2k+d-1$ to the $2k+1$ near 1-factors (one label for each near 1-factor and make sure that the near 1-factor with 0 as the isolated vertex will receive the label $d-1$. Then we are done as this labelling will not be in conflict with the labels assigned to the edges in M_i's or the vertices.
- Otherwise, take a near 1-factorization $\{NF_1, NF_2, ..., NF_{2k+1}\}$ of K_{2k+1} as in Lemma 3.2. First, we use the integers of $[d-1, 2k+d-1]$ to label them : for $2 \leq i \leq 2k-d+2$, The edges of NF_i are labelled $2k-i+1$. For the rest, we divide it into two subcases.

The strategy of labelling is the same as case 1 and here we will give the labelling and omit the verification.

case 2.1 : d is even. The edges of NF_1 are labelled $2k+d-2$ and for $0 \leq i \leq (d-4)/2$, assign the label $2k+d-4-2i$ to the edges of NF_{2k-2i}. Then assign the rest labels to the remaining near 1-factors (one label each).

We now adjust the labels for a few edges in order to achieve a valid labelling.

(a) For $3 \leq i \leq 2k-d+2$, in NF_i, the label of the edges which have both endvertices in the set $\{0\} \cup (\cup_{j=d}^{d+i-3}\{-j\}$ is changed to $-(2d+i-3)$. Recall that the original labels for all these edges were: $2k-2, 2k-3, ..., d-1$.

(b1) If $k \geq d$, for $0 \leq i \leq (d-4)/2$, in NF_{2k-2i}, reassign the label of $-(2d-2-2i)$ to the edge $(0, -(2k+d-1-i))$.

(b2) If $k < d$, for $0 \leq i \leq k-d/2-1$, in NF_{2k-2i}, reassign the label $-(2k-2-2i)$ to the edge $(0, -(2k+d-1-i))$.

case 2.2 : d is odd. In this case, the edges of NF_1 are labelled $2k+d-1$, and for $0 \leq i \leq (d-3)/2$, the edges of NF_{2k-2i} are labelled $2k+d-3-2i$.

(a) For $3 \leq i \leq 2k-d+2$, in NF_i, the label of the edges which have both endvertices in the set $\{0\} \cup (\cup_{j=d}^{d+i-3}\{-j\}$ is changed to $-(2d+i-3)$.

(b1) If $k \geq d$, for $0 \leq i \leq (d-3)/2$, in NF_{2k-2i}, reassign the label $-(2d-2-2i)$ to the edge $(0, -(2k+d-1-i))$.

(b2) If $k < d$, then for $0 \leq i \leq k-(d+1)/2$, in NF_{2k-2i}, reassign the label $-(2k-2-2i)$ to the edge $((0, -(2k+d-1-i))$.

■

Lemma 3.1 *There exists a 1-factorization $\{F_1, F_2, ..., F_{2k+1}\}$ of K_{2k+2} with a vertex set $\{0\} \cup [-d-2k, -d]$ such that it satisfies the following properties:*

(a) If i is even, F_i has $i/2$ edges covering the vertices $\{-d, -d-1, \ldots, -d-i+1\}$, if $i \leq 2k$,

(b) If i is odd, F_i has $(i+1)/2$ edges covering the vertices $\{0, -d, ..., -d-i+1\}$, if $i \leq 2k+1$.

Proof. We give an explicit construction of such a 1-factorization. Let $f_i = \{(0, -d-i), (-d-1-i, -d-2k-i), (-d-2-i, -d-2k+1-i), \ldots, (-d-k-i, -d-k-1-i)\}$ for $0 \leq i \leq 2k$. This is a standard cyclic 1-factorization of K_{2k+2}. Now we define F_i as follows.

Let $F_{2i-1} = f_{i-1}$ and $F_{2i} = f_{k+i}$, for $1 \leq i \leq k$ and $F_{2k+1} = f_k$. We can check that both conditions are satisfied. ■

Lemma 3.2 *There exists a near 1-factorization $\{NF_1, NF_2, \ldots, NF_{2k+1}\}$ of K_{2k+1} with a vertex set $\{0\} \cup [-d-2k+1, -d]$ such that it satisfies the following properties:*

(a) *If $i \geq 4$ is even, NF_i has $i/2-1$ edges covering the vertex set $\{-d, -d-1, \ldots, -d-i/2-1\}$ and the vertex $-i/2-d$ is not covered by NF_i. NF_2 has $-d$ as the isolated vertex.*

(b) *If i is odd, NF_i has $(i-1)/2$ edges covering the vertex set $\{0, -d, \ldots, -d-(i-1)/2\}$ and the vertex $-(i+1)/2-d$ is not covered by NF_i.*

Proof. This near 1-factorization can be obtained by deleting the vertex $-d$ from the 1-factorization in Lemma 1 and then relabel the vertex $-d-i$ by $-d-i+1$, for $1 \leq d \leq 2k$. ■

Corollary 3.1 *If n is odd then $\lambda_d(K_n) = \min\{n+2d-3, 2n+d-3\}$.*

3.2 Even complete graphs

Theorem 3.2 *If n is even and $n > 6d^2 - 10d + 4$, then $\lambda_d^T(K_n) = n + 2d - 2$.*

Proof. By Proposition 2.4, $\lambda_d^T(K_n) \leq n + 2d - 2$.

Let G be a graph on n vertices. Suppose that G admits a $(d,1)$-total labelling with labels in $[0, n+2d-3]$. Then each label l induces a matching M_l over the edges of G. Moreover, this label is not adjacent to the vertices with labels in $[l-d+1, l+d-1]$. Let $b(l)$ be the number of labels in $I_l = [l-d+1, l+d-1]$ that are assigned to no vertex. Then M_l contains at most $\left\lfloor \frac{n-2d+1+b(l)}{2} \right\rfloor = \frac{n-2d}{2} + \left\lceil \frac{b(l)}{2} \right\rceil$ edges and G contains at most $(n+2d-2)\frac{n-2d}{2} + \sum_{i=0}^{n+2d-3} \left\lceil \frac{b(l)}{2} \right\rceil$ edges. Each non assigned label is contained in $2d-1$ intervals I_l. And for $1 \leq i \leq d-1$ the labels $-i$ and $n+2d-3+i$ are contained in $d-i$ intervals I_l. Hence $\sum_{i=0}^{n+2d-3} b(l) \leq (2d-2)(2d-1) + 2\sum_{i=1}^{d-1} i = 5d^2 - 7d + 2$. Since $\sum_{i=0}^{n+2d-3} \left\lceil \frac{b(l)}{2} \right\rceil \leq \sum_{i=0}^{n+2d-3} b(l)$, if $n > 6d^2 - 10d + 4$, then G has less than $n(n-1)/2$ edges. Thus G is not complete. ■

If $d = 1$, then $6d^2 - 10d + 4 = 0$. Hence as a corollary, we have the result of Bezhad, Chartand and Cooper [2] on total colouring :

Corollary 3.2 *$\chi^T(K_n) = \lambda_1^T(K_n) + 1$ equals n if n is odd, and $n+1$ if n is even.*

Proposition 3.2 *Let n be an even integer greater than 4. If $d \geq n-3$, then $\lambda_d^T(K_n) \leq n + 2d - 3$.*

Proof. Let us first prove that $\lambda_{n-3}^T(K_n) \leq 3n - 9$. Label the vertices with $\{0, 1, 2n-7\} \cup [2n-5, 3n-9]$. Since $n > 4$ then $2n - 7 > 1$, thus the vertices receive different labels. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-7\} \cup [2n-5, 3n-9]$ with $[0, n-4]$. It is possible since $\chi'(K_{n-2}) = n-3$. For $j \in [2n-5, 3n-9]$, label the edge $(1, j)$ with $j - n + 3$ and the edge $(0, j)$ with $j - n + 2$. Complete the labelling by assigning $3n-10$ to $(0, 2n-7)$, $3n-9$ to $(1, 2n-7)$, and $3n-8$ to $(0, 1)$. One can check that this is a valid $(n-3,1)$-total labelling of K_n. To obtain a $(n-3+i,1)$-total labelling start from the above labelling and change the label l by $l+i$ if it is in $[n-5,2n-6]$ and $l+2i$ if it is in $[2n-5,3n-9]$. ■

Proposition 3.3 *i)* $\lambda_2^T(K_4)=6$

ii) $\lambda_{3+i}^T(K_4) \le 7+2i$. *In particular,* $\lambda_3^T(K_4)=7$ *and* $\lambda_4^T(K_4)=9$.

i) By Proposition 2.4, there is a (2,1)-total labelling l of K_4 with span 6.

Suppose that there exists an (2,1)-total labelling l of K_4 in $[0,5]$. For any vertex v, let $A(v)$ be the set of labels of its three incident edges. Now since each vertex must receive a different label, there are two vertices u and v such that $l(u)+1=l(v)$. Clearly, $|A(u) \cap A(v)| \ge 2$ since $l(u)$ and $l(u+1)$ are not contained in both $A(u)$ and $A(v)$. Hence two edges share the same label l. Necessarily, there is no vertex labelled l, $l-1$ and $l+1$. Since only two labels are not assigned to vertices, either $l=0$ and the four vertices are labelled 2, 3, 4 and 5 or symmetrically $l=5$ and the four vertices are labelled 0, 1, 2 and 3. This implies that only five edges may be labelled which is a contradiction. Indeed in the first case, the label 0 may be assigned to two edges, the labels 1, 2 and 5 to one edge and 3 and 4 to none.

ii) A $(3+i)$-total labelling in $[0,7+2i]$ is given by the following adjacency matrix :

	0	$4+i$	$6+2i$	$7+2i$
0		$7+2i$	$3+i$	$4+i$
$4+i$	$7+2i$		1	0
$6+2i$	$3+i$	1		2
$7+2i$	$4+i$	0	2	

By Proposition 3.1, $\lambda_3^T(K_4) \ge 7$ and $\lambda_4^T(K_4) \ge 9$. So $\lambda_3^T(K_4)=7$ and $\lambda_4^T(K_4)=9$. ■

Proposition 3.4 *Let n be an even integer greater than 5. Then* $\lambda_{n-4}^T(K_n)=3n-11$.

Proof. By Proposition 3.1, $\lambda_{n-4}^T(K_n) \ge 3n-11$.

Let us now show an $(n-4,1)$-total labelling of K_n in $[0,3n-11]$. Label the vertices with $\{0,1,2n-9\} \cup [2n-7,3n-11]$. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-9\} \cup [2n-7,3n-11]$ with $[0,n-4]$ in such a way that the edge $e=(2n-7,2n-9)$ is labelled $n-4$. Its is possible since $\chi'(K_{n-2})=n-3$. The label of e is not valid. Change it to $3n-11$. For $j \in [2n-7,3n-11]$, label the edge $(1,j)$ with $j-n+4$ and the edge $(0,j)$ with $j-n+3$. Complete the labelling by assigning $3n-13$ to $(0,2n-9)$, $3n-12$ to $(1,2n-9)$, and $3n-11$ to $(0,1)$. ■

Proposition 3.5 *Let n be an even integer greater than 7. Then* $\lambda_{n-5}^T(K_n)=3n-13$.

Proof. By Proposition 3.1, $\lambda_{n-5}^T(K_n) \ge 3n-13$.

Let us now show an $(n-5,1)$-total labelling of K_n in $[0,3n-13]$. Label the vertices with $\{0,1,2n-11\} \cup [2n-9,3n-13]$. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-11\} \cup [2n-9,3n-11]$ with $[0,n-4]$ in such a way that the edges $e_1 = (2n-11, 2n-9)$ and $e_2 = (2n-11, 2n-8)$ are labelled $n-4$ and $n-5$. The labels of e_1 and e_2 are not valid. Change them to $3n-14$ and $3n-13$ respectively. For $j \in [2n-7, 3n-13]$, label the edge $(1,j)$ with $j-n+5$ and the edge $(0,j)$ with $j-n+4$. Complete the labelling by assigning $n-3$ to $(1,2n-8)$, $n-5$ to $(0,2n-8)$, $n-4$ to $(1,2n-9)$, $3n-13$ to $(0,2n-9)$, $3n-15$ to $(1,2n-11)$, $3n-16$ to $(0,2n-11)$, and $3n-14$ to $(0,1)$. By construction, the labels of incident edge and vertex are at distance at least $n-5$. And adjacent edges have different labels if $3n-15 > 2n-8$ that is $n > 7$. ■

Proposition 3.6 *Let n be an even integer greater than 7. Then $\lambda^T_{n-6}(K_n) = 3n-15$.*

Proof. By Proposition 3.1, $\lambda^T_{n-6}(K_n) \geq 3n-15$.

We give an $(n-6,1)$-total labelling of K_n in $[0, 3n-15]$ as follows. Label the vertices with $\{0,1,2,3,2n-11\} \cup [2n-9, 3n-15]$. Label the edges of the complete subgraph induced by the vertices labelled in $\{2n-11\} \cup [2n-9, 3n-15]$ with $[0, n-6]$. For $j \in [2n-9, 3n-15]$, label the edge $(3,j)$ with $j-n+6$, the edge $(2,j)$ with $j-n+5$ the edge $(1,j)$ with $j-n+4$ and the edge $(0,j)$ with $j-n+3$. Change the label of $(0,2n-9)$ into $3n-15$ and label $(0,2n-11)$ with $n-6$. Complete the labelling by the following labelling of the complete induced by $\{0,1,2,3,2n-11\}$.

	0	1	2	3	$2n-11$
0		$3n-18$	$3n-17$	$3n-16$	$n-6$
1	$3n-18$		$3n-15$	$2n-12$	$3n-17$
2	$3n-17$	$3n-15$		$2n-11$	$3n-16$
3	$3n-16$	$2n-12$	$2n-11$		$3n-15$
$2n-11$	$n-6$	$3n-17$	$3n-16$	$3n-15$	

■

Problem 1 What is $\lambda^T_d(K_n)$ when $d+6 \leq n \leq 6d^2-10d+4$ and n even? $n+2d-3$ or $n+2d-2$?

4 (d,1)-total labelling of graph with small maximum degree

4.1 $\Delta = 2$

Theorem 4.1 *Let G be a connected graph with maximal degree 2.*

(i) $\lambda^T_2(G) = 4$.

(ii) Let $d \geq 3$. If G is an odd cycle then $\lambda^T_d(G) = d+3$ otherwise $\lambda^T_d(G) = d+2$.

Proof. (i) By Proposition 2.1 $\lambda^T_2(G) \geq 4$. If G is bipartite, then by Corollary 2.1, $\lambda^T_2(G) \leq 4$. Suppose now that G is not bipartite, then it is an odd cycle $(a_0, a_1, a_2, \ldots, a_{2p}, a_0)$. Then a $(2,1)$-total labelling $l$of G is the following : For $i=1$ to p, $l(a_{2i-1}a_{2i})=4$ and $l(a_{2i})=0$; for $i=1$ to $p-1$, $l(a_{2i+1})=1$ and $l(a_{2i}a_{2i+1})=3$; and $l(a_0)=4$, $l(a_1)=2=l(a_{2p}a_0)=2$ and $l(a_0a_1)=0$.

(ii) If G is not an odd cycle it is bipartite then by Corollary 2.1 and Proposition 2.1, $\lambda_d^T(G)=d+2$. Suppose now that G is an odd cycle $(a_0,a_1,a_2,\dots,a_{2p},a_0)$. By Proposition 2.2, $\lambda_2^T(G)\leq d+3$. Suppose now that G admits a $(d,1)$-total labelling in $[0,d+2]$. Then vertices must be labelled with 0, 1, $d+1$ or $d+2$. Since an odd cycle is not 3-colourable, there must be an edge whose endvertices are labelled with one label in $\{0,1\}$ and one in $\{d+1,d+2\}$. Now since $d+2<2d$ this edge may not be labelled. Contradiction. ■

4.2 $\Delta = 3$

Theorem 4.2 *If* $\Delta(G)\leq 3$ *then* $\lambda_2^T(G)\leq 6$.

Proof. If $G=K_4$, then we have the result by Proposition 3.3. So we may suppose that G is not complete. Then by Brook's theorem, $\chi(G)=3$ and G is tripartite. Let (X,Y,Z) be a tripartition of $V(G)$ such that for each $x\in X$, $N(x)\cap Y\neq\emptyset$ and $N(x)\cap Z\neq\emptyset$, and for each $y\in Y$, $N(y)\cap Z\neq\emptyset$.

We will now construct a (2,1)-total labelling of G in three steps :

1) First assign the label 0, 1, 2 respectively to the vertices of X, Y and Z.

2) Consider H' the graph induced by the edges joining vertices of Z to vertices of $X\cup Y$. It is bipartite and $\Delta(H')\leq 3$. Thus, by König's theorem, we may label its edges with the three labels 4, 5 and 6.

3) Now consider H the graph induced by the edges joining a vertex of X to a vertex of Y. By definition of the tripartition, H is bipartite and $\Delta(H)\leq 2$. Hence, it is the union of even cycle and paths.

    a) Let us first label the (even) cycles. Let $C=(a_1,a_2,b_1,b_2,\dots,a_p,b_p,a_1)$ be a cycle of H. For $1\leq i\leq p$, assign the label 3 to each edge a_ib_i and label the edge b_ia_{i+1} with the label in $\{4,5,6\}$ which is not used by the two edges joining b_i to a vertex of Z and a_{i+1} to a vertex of Z.

    b) In the same way as in a), label each odd paths $(a_1,a_2,b_1,b_2,\dots,a_p,b_p)$.

    c) Let us now label the even paths one after another.
    Let $P=(a_1,a_2,b_1,b_2,\dots,a_p,b_p,a_{p+1})$ be a yet unlabelled even path. For $1\leq i\leq p$, assign the label 3 to each edge a_ib_i and for $1\leq i\leq p-1$, assign to the edge b_ia_{i+1} with the label in $\{4,5,6\}$ which is not used by the two edges joining b_i to a vertex of Z and a_{i+1} to a vertex of Z. The only edge that remains to be labelled is $e=b_pa_{p+1}$. Therefore, we may need to relabel the vertices b_p and a_{p+1} and the formerly labelled edge b_pz_0 where $z_0\in Z$. Let z_1 and z_2 be the two neighbours of a_{p+1} in Z.

(i) If there is a label $l \in \{4,5,6\}$ which is not used to label $e_0 = b_p z_0$, $e_1 = a_{p+1} z_1$ or $e_2 = a_{p+1} z_2$, then assign l to $b_p a_{p+1}$.
(ii) If for some $i \in \{0,1,2\}$, one of the edges incident to z_i is labelled 0 then one can relabel e_i with a new label in $\{4,5,6\}$ and assign the old one to e.
(iii) If for all $i \in \{0,1,2\}$, no edge incident to z_i is labelled 0, then do the following : If e_0 is labelled 4, then relabel e_0 with 0, b_p with 5 and a_{p+1} with 3 and label e with 1. If not without loss of generality, e_1 is labelled 4. Then relabel e_0 and e_1 with 0, b_p with 5 and a_{p+1} with 3 and label e with 1.

■

Theorem 4.2 is tight since $\lambda_2^T(K_4) = 6$ by Proposition 3.3. However, we think that it is the only graph with $\Delta = 3$ and $\lambda_2^T = 6$:

Conjecture 4.1 *If $\Delta(G) \leq 3$ and $G \neq K_4$ then $\lambda_2^T(G) \leq 5$.*

Theorem 4.2 and Proposition 2.2 imply that Conjecture 2.1 holds when $\Delta = 3$. In particular, we get that $\lambda_3^T(G) \leq 8$. This upper bound is not best possible :

Theorem 4.3 *If $\Delta(G) \leq 3$ then $\lambda_3^T(G) \leq 7$.*

Proof. If $G = K_4$, then we have the result by Proposition 3.3. So we may suppose that G is not complete. Then by Brook's theorem, $\chi(G) = 3$ and G is tripartite. Let (X, Y, Z) be a tripartition of $V(G)$ such that for each $x \in X$, $N(x) \cap Y \neq \emptyset$ and $N(x) \cap Z \neq \emptyset$, and for each $y \in Y$, $N(y) \cap Z \neq \emptyset$.

Let H be the bipartite graph induced by $X \cup Y$ and H' the graph induced by the edges joining vertices of Z to vertices of $X \cup Y$. The graph H has maximum degree at most 2, so its components are paths and (even) cycles. The graph H' is bipartite and $\Delta(H') \leq 3$. Thus, by König's theorem, it is 3-edge colourable. Let $\mathcal{C}$ be the set of edge colourings of H' with colours 5, 6 and 7.

The *ends* of the path $P = (a_1, a_2, \ldots, a_n)$ are the edges $(a_1, a_2]$ and $[a_{n-1}, a_n)$. The different brackets are used to distinguish the endvertices.

Let $c \in \mathcal{C}$ and let $(x, y]$ be an end of an even path of H. Let e_0 be the edge of H' incident to y and e_1 and e_2 the edges of H' incident to x. We say that $(x, y]$ is *c-good* if $\{c(e_0), c(e_1), c(e_2)\} \neq \{5,6,7\}$ or $c(e_0) = 5$. An end that is not c-good is said to be *c-bad*. A component of H is *c-bad* if it is an even path (with length at least 2) with two c-bad ends.

Let us now consider the edge colouring $c_0 \in \mathcal{C}$ that minimizes the number of bad components in H. Let us prove that c_0 has no bad paths. Suppose for contradiction that there is a bad path P_0. Let $(x_0, y_0]$ be one of its end and a the colour labelling the edge of H' incident to y_0. Since $(x_0, y_0]$ is bad $a \neq 5$ and an edge incident to x_0 is labelled 5. Let Q_0 be the longest path of H' starting at x with alternating colours 5 and a. Let c_1 be the edge colouring obtained from c_0 by interchanging the colours a and 5 along Q_0. Let z_0 be the endvertex of Q_0 distinct from x_0. Since c_0 also minimizes the number of bad components in H, then edge colouring c_1 also minimizes the number of bad components in H. Moreover, P_0

is c_1-good thus P_1, the component of z_0 in H, must be c_1-bad and have been c_0-good. This implies that P_1 is an even path and that z_0 belongs to an end of $(x_1, y_1]$ of P_1. Furthermore if $z_0 = y_1$, then $c_0(z_0) \neq a$ otherwise $c_1(z_0) = 5$ and P_1 is c_1-good. In particular, $z_0 \neq y_0$. And because $(x_0, y_0]$ is c_1-good, $P_0 \neq P_1$. Set $t_1 = \{x_1, y_1\} \setminus \{z_0\}$. Since $(x_1, y_1]$ is c_1-bad, t_1 is adjacent to an edge e_1 labelled with a or 5. Let Q_1 be the longest path of H' starting at t_1 with alternating colours 5 and a. Let z_1 be the endvertex of Q_1 distinct from x_1, P_2 the component of z_1 in H and c_2 the edge colouring obtained from c_1 by interchanging the colours a and 5 along Q_1. As before, c_2 minimizes the number of bad components and P_2 is c_2-bad. And $z_1 \neq y_0$. Thus $P_2 \neq P_0$ and because z_1 is not in $\{x_1, y_1\}$, $P_2 \neq P_1$. And so on by induction, for any $i \geq 0$ one constructs i distinct components of H. This is a contradiction since G is finite.

Hence c_0 has no bad components.

We will now construct (2,1)-total labelling of G from c_0. First assign the label 0, 1, 2 respectively to the vertices of X, Y and Z. And label the edges of H' according to c_0. Let us now label the components of H. Let C be such a component.

a) If C is a cycle $(a_1, a_2, b_1, b_2, \ldots, a_p, b_p, a_1)$. For $1 \leq i \leq p$, assign the label 4 to each edge $a_i b_i$ and label the edge $b_i a_{i+1}$ with the label in $\{5, 6, 7\}$ which is not used by the two edges joining b_i to a vertex of Z and a_{i+1} to a vertex of Z.

b) Proceed analogously if C is an odd path $(a_1, a_2, b_1, b_2, \ldots, a_p, b_p)$.

c) Suppose now that C is the even path $(a_1, a_2, b_1, b_2, \ldots, a_p, b_p, a_{p+1})$. By symmetry, we may suppose that $[b_p, a_{p+1})$ is good.

   For $1 \leq i \leq p$, assign the label 4 to each edge $a_i b_i$ and for $1 \leq i \leq p-1$, assign to the edge $b_i a_{i+1}$ with the label in $\{5, 6, 7\}$ which is not used by the two edges joining b_i to a vertex of Z and a_{i+1} to a vertex of Z.

   Let z_0 be the neighbour of b_p in Z and z_1 and z_2 be the two neighbours of a_{p+1} in Z. If there is a label $l \in \{4, 5, 6\}$ which is not used to label $e_0 = b_p z_0$, $e_1 = a_{p+1} z_1$ or $e_2 = a_{p+1} z_2$, then assign l to $b_p a_{p+1}$.

   Otherwise since $[b_p, a_{p+1})$ is good, e_0 is labelled 5 and e_1 and e_2 are labelled with 6 and 7. Then relabel z_0 with 3, b_p with 7, a_{p+1} with 0 and e_0 with 0 and label $b_p a_{p+1}$ with 3.

By construction, this is a $(3, 1)$-total labelling of G. ∎

4.3 $\Delta = 4$

Theorem 4.4 *If* $\Delta(G) \leq 4$ *then* $\lambda_2^T(G) \leq 8$.

Proof. If G is K_5 then we have the result by Theorem 3.1. So, by Brook's theorem, we may suppose that G is 4-colorable and therefore 4-partite. Let (A, B, C, D) be a 4-partition of G such that $G(A, B)$ and $G(C, D)$ are bipartite with max degree at most two. We will now construct a (2,1)-total labelling of G with the labels in $[0, 8]$.

I) $G(A \cup B, C \cup D)$ has degree at most 4. So by König's theorem, we may label its edges with the four labels 5, 6, 7 and 8.

II) Label every isolated vertex in $G(A, B)$ with 0 and the isolated vertices in $G(C, D)$ with 2.

III) Let us now label the edges and the vertices of the graph H induced by the non isolated vertices of $G(C, D)$. Note that H is the union of even cycles and paths.

Let xy be an edge of $G(C, D)$ with x and y of degree two in $G(C, D)$. If y is labelled 3, a *usual extension* to xy is defined as follows :

- If one label $l \in \{5, 6, 7, 8\}$ does not label any edge incident to x and y then label xy with l and x with 2.

Otherwise, the edges adjacent to xy have distinct labels.

- If no edge incident to x is labelled 5 then assign 4 to x and 1 to xy.

- If an edge of incident to x is labelled 5, then relabel it with 4 and assign 2 to x and 5 to xy.

A) First label the cycles. Let $(u_1, v_1, \dots, u_p, v_p, u_1)$ be a cycle. For $1 \leq i \leq p$ assign the label 3 to each v_i and 0 to each edge $v_i u_{i+1}$. And apply the usual extension to each $u_i v_i$.

B) Label now the odd paths. Let $(v_0, u_1, v_1, \dots, v_{p-1}, u_p)$ be an odd path. In the same way as for cycles, we assign labels to its vertices and edges.

C) Consider now the even paths $P_1, P_2, \dots, P_m$. For $1 \leq j \leq m$, let Q_j be the path obtained from P_j by deleting its final edge (that is if $P_j = (v_0, u_1, v_1, \dots, v_{p-1}, u_p, v_p)$ then $Q_j = (v_0, u_1, v_1, \dots, v_{p-1}, u_p)$.

We first assign labels to all the Q_j with the following method. For $1 \leq i \leq p-1$ assign the label 0 to each edge $v_i u_{i+1}$. If no edge adjacent to v_p is labelled 5 then assign 3 to each u_i $1 \leq i \leq p$ otherwise assign 3 to every v_i, $1 \leq i \leq p+1$. Apply the usual extension to each $u_i v_i$, $1 \leq i \leq p-1$.

Let us now label the final edge (i.e. $u_p v_p$) of every Q_j and its non-labelled vertex.

1) If no edge incident to v_p is labelled 5 then label v_p with 4 and assign 1 to $u_p v_p$.

Otherwise the vertex v_p is labelled 3 and the label of u_p is unlabelled.

2) If one label $l \in \{5, 6, 7, 8\}$ is not used for any edge incident to u_p and v_p then label u_p with 2 and label $u_p v_p$ with l.
3) If no edge incident to u_p is labelled 5 then label u_p with 4 and assign 1 to $u_p v_p$.
4) If the edges incident to u_p are labelled 5 and 6, then label u_p with 8 and assign 1 to $u_p v_p$.

5) Suppose that the two already labelled edges incident to u_p, u_pa and u_pb are labelled 5 and l_1 respectively, and that the edges adjacent to v_p are labelled 5, 6 and l_2 with $\{l_1, l_2\} = \{7, 8\}$.

   Let l be the label of $[4, 8]$ which is not used for any edge incident to b.

   If $l \neq 5$, then relabel u_pb with l, assign 2 to u_p and l_1 to u_pv_p. If $l = 5$, then b is an isolated vertex in $G(A, B)$. Hence b is labelled 0. Relabel u_pb with 2 and assign l_1 to u_p and 1 to u_pv_p.

IV) Let us now label the edges and the vertices of the graph H' induced by the non isolated vertices of $G(A, B)$. Note that H' is the union of even cycles and paths. By the previous construction of the labelling of $G(C, D)$ every edge from a vertex of H' to a vertex of $C \cup D$ has a label in $[4, 8]$. Moreover no vertex of $C \cup D$ has been labelled 0 or 1.

Let $(u_1, v_1, \ldots, u_p, v_p, u_1)$ be a cycle. For $1 \leq i \leq p$, assign the label 1 to v_i, the label 0 to u_i, and 3 to v_iu_{i+1}. We now need to label the edges u_iv_i. Since u_i and v_i are joined to at most two vertices each in $C \cup D$ then there is label $l \in [4, 8]$ that is assigned to no edge incident to u_i or v_i. Then assign l to u_iv_i.

In the same way, we may label an odd path $(v_0, u_1, v_1, \ldots, v_{p-1}, u_p)$.

Let us consider an even path $(v_0, u_1, v_1, \ldots, v_{p-1}, u_p, v_p)$. Let us label the edges and vertices of $(v_0, u_1, v_1, \ldots, v_{p-1}, u_p)$ as in an odd path. And assign 1 to v_p. If there is a label $l \in [4, 8]$ that is assigned to no edge incident to u_p or v_p, then assign l to u_pv_p. Otherwise, there is a unique edge e adjacent to u_pv_p that is labelled 4. By the labelling of $G(C, D)$ this edge is incident to a vertex w labelled 2 and adjacent to an edge labelled 0. And there is a label l of $[5, 8]$ which is not assigned to an edge adjacent to w. Then relabel e with l and label u_pv_p with 4.

■

5 Extending a vertex colouring into a $(d, 1)$-total labelling

One approach to prove the Conjecture 2.1 is to obtain a small function $a(d)$ such that a $\Delta + a(d)$ $(d, 1)$-total labelling of a graph can be constructed by extending a vertex colouring with a suitable edge colouring.

Conjecture 5.1 *Let $d \geq 1$. There is an integer $a(d)$, such that for any vertex colouring c_v of a non-complete graph G with colours in $[0, \Delta - 1]$, there is an edge colouring c_e of G with colours in $[0, \Delta + a(d)]$ such that $c_v \cup c_e$ is a $(d, 1)$-total labelling of G.*

Remark 5.1 The List Colouring Conjecture implies that Conjecture 5.1 is true for $a(d) = 4d - 2$.

Conjecture 5.2 (List Colouring Conjecture) The chromatic index is equal to the list chromatic index, that is $\chi' = \chi'_l$.

Since every graph is $(\Delta+1)$-edge colourable (Vizing's Theorem), this conjecture implies that it also is $(\Delta+1)$-edge choosable.

Let c_v be a vertex colouring of a non-complete graph with colours in $[0, \Delta-1]$. For any edge $e=(x,y)$, there is a set $L(e) \subset [0, \Delta+4d-2]$ of $\Delta+1$ colours such that $L(e) \cap ([c_v(x)-d+1, c_v(x)+d-1] \cup [c_v(y)-d+1, c_v(y)+d-1] = \emptyset$. Then since G is $\Delta+1$-choosable, there exists a desired edge colouring.

5.1 Some probabilistic results

In [5], McDiarmid and Reed proved that a graph G with n vertices $\chi^T(G) \le \chi'(G) + k + 1$ where k is an integer such that $k! > n$. Analogously, one can prove the following generalisation to $(d,1)$-labelling.

Theorem 5.1 *If G is a graph with n vertices and k is an integer with $\frac{k!}{(2d-1)^k} > n$ then $\lambda_d^T(G) \le \chi'(G) + k + 3d - 3$.*

Proof. We may assume that G is not complete. Let $q = \chi'(G)$. By Brook's Theorem, G has a vertex colouring c using the q colours of $[0, q-1]$. Let $\mathcal{M} = \{M_1, M_2, \dots, M_q\}$ be the collection of matchings in an edge colouring of G using q colours.

Given a bijection π from $\mathcal{M}$ to $[0, q-1]$, let the "rejection graph" G_π be the subgraph of G containing those edges xy such that $\pi(xy) \in [c(x)-d+1, c(x)+d-1] \cup [c(y)-d+1, c(y)+d-1]$. Then clearly,

$$\lambda_d^T(G) \le q + \chi'(G_\pi) + d - 2 \le q + \Delta(G_\pi) + d - 1.$$

We shall prove that for some bijection π, we have $\Delta(G_\pi) \le k+2$, by considering a random bijection with all $q!$ equally likely.

Consider a vertex $v \in G$, with set N of at least $k+2d-1$ neighbours in G. Let $\mathcal{N}$ be the collection of sets $N' \subset N$ with cardinality k. Also, for each set $N' \in \mathcal{N}$, let $A(N')$ be the event that for each vertex $w \in N'$, the matching M containing the edge vw is mapped to a colour in $[c(w)-d+1, c(w)+d-1]$. Clearly, we have:

$$Pr(A(N')) \le \frac{(2d-1)^k}{\prod_{i=0}^{k-1}(q-i)} = \frac{(q-k)!}{q!}(2d-1)^k.$$

Let $d_\pi(v)$ denote the degree of v in G_π. If $d_\pi(v) \ge k+2d-1$ then the event $A(N')$ must occur for at least one $N' \in \mathcal{N}$. And $|\mathcal{N}| = \binom{|N|}{k}$ so

$$Pr\{d_\pi(v) \ge k+2d-1\} \le \binom{|N|}{k}\frac{(q-k)!}{q!}(2d-1)^k.$$

Since $|N| \le \Delta$ and $q \ge \Delta$ then

$$Pr\{d_\pi(v) \ge k+2d-1\} \le \binom{\Delta}{k}\frac{(\Delta-k)!}{\Delta!}2d-1^k = \frac{(2d-1)^k}{k!}.$$

Now we obtain

$$Pr\{\Delta(G_\pi) \geq k + 2d - 1\} \leq n\frac{(2d-1)^k}{k!} < 1$$

because $\frac{k!}{(2d-1)^k} > n$. Hence, for some bijection π, $\Delta(G_\pi) \leq k + 2d - 2$ as required. ■

Corollary 5.1 *As $n \to \infty$, $\lambda_d^T(G) \leq \chi'(G) + O(\log n / \log\log n)$.*

Following the proof of the Theorem 2.1 of [5], one can prove the following result asserting that as $n \to \infty$ the proportion of graphs on vertices $1, 2, \ldots, n$ with total $(d, 1)$-labelling number $\lambda_d^T > \Delta + 2d - 1$ is very small:

Theorem 5.2 *Let p and c be constant with $0 < p < 1$ and $0 < c < \min\{\frac{1}{3}, \frac{p}{2}\}$. Then*

$$P\{\lambda_d^T(G_{n,p}) > \Delta + 2d - 1\} = o(n^{-cn/2}).$$

5.2 Extending with a fractionnal edge colouring

One can relax the constraints and try to extend the vertex colouring with a fractionnal edge colouring.

Let $\mathcal{M}$ be the set of matchings of G. Given a vertex colouring c with coulours in $[1, \Delta-1]$.

We want to minimize the *fractionnal extend span* $\Delta + d - 2 + \sum_{M\in\mathcal{M}} w_{\Delta+d-1}(M)$ under the following constraints :

- for $0 \leq i \leq \Delta + d - 2$, $\sum_{M\in\mathcal{M}} w_i(M) \leq 1$.
  Each already used colours has a weight at most one on each edge.
- for $e \in E(G)$, $\sum_{e\in M} \sum_{i\in P(e)} w_i(M) \geq 1$,
  where $P(e) = [0, \Delta + d - 1] \setminus ([c(x) - d + 1, c(x) + d - 1] \cup [c(y) - d + 1, c(y) + d - 1])$.
  Each edge must be covered by a weight of one by allowed matching (i.e. with colours at least two apart from the colours of its vertices).

Theorem 5.3 *Let G be a (non-complete) graph. For any vertex colouring c of G with colours in $[0, \Delta - 1]$, the fractionnal extend span is at most $\Delta + 3d$.*

Proof. Let $M_0, M_1, \ldots, M_\Delta$ be the matching of a $\Delta+1$ edge colouring of G. For $0 \leq j \leq \Delta$, set $w_i(M_j) = \frac{1}{\Delta+1}$ for $0 \leq i \leq \Delta + d - 2$ and and set $w_{\Delta+d-1}(M_j) = \frac{3d}{\Delta+1}$.

Let us prove that the two constraints are fullfilled:

For $0 \leq i \leq \Delta + d - 2$, we have :

$$\sum_{M\in\mathcal{M}} w_i(M) = \sum_j w_i(M_j) = (\Delta + 1)\frac{1}{\Delta + 1} = 1$$

.

Let e be an edge, it is contained in one matching M_{j_e}.

$$\sum_{e\in\mathcal{M}}\sum_{i\in P(e)} w_i(M) = \sum_{i\in P(e)} w_i(M_{j_e}) = \frac{3d}{\Delta+1} + \frac{1}{\Delta+1}(|P(e)|-1) \geq \frac{3d}{\Delta+1} + \frac{\Delta-3d+1}{\Delta+1} \geq 1$$

Then the fractionnal extend span is at most :

$$\Delta + \sum_{M\in\mathcal{M}} w_{\Delta+1}(M) = \Delta + \sum_{j}\frac{3d}{\Delta+1} = \Delta + 3d$$

■

6 Conclusion

In this paper, we have given a number of evidences for Conjecture 2.1 to be true. Note that this conjecture implies that $\lambda_d^T(G) \leq (1+a)\Delta + (2-a)d - 1$ for any $0 \leq a \leq 1$. Proposition 2.2 aserts this for $a = 1$ and it is exactly Conjecture 2.1 if $a = 0$. It would be interesting to prove some intermediate results by showing this inequality for some $a < 1$. For example, if $\Delta = 4$, it holds for $a = 1/2$ according to Theorem 4.4.

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